# Book Review

*Vicious Circles: On the Mathematics of Non-Wellfounded Phenomena, Jon Barwise and Lawrence Moss*

Varol Akman
*Bilkent University, Ankara, Turkey*

Vicious Circles: On the Mathematics of Non-Wellfounded Phenomena
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"[T]o a greater or lesser degree, every scientific advance marks some departure from the common sense that preceded it." These words of Irving M. Copi (Copi, 1979, p. 195) apparently summarize the nature of *Vicious Circles* in the most concise fashion. Following the steps of Aczel's ground-breaking monograph (Aczel, 1988) which marked a departure from the 'common sense' that is attributed to classical set theory, this new book (abbreviated as VC in the sequel) of Jon Barwise and Larry Moss not only offers an introduction to the revolutionary and fascinating topic of non-wellfounded sets (a.k.a. hypersets) but also becomes the most authoritative source for any serious researcher (mathematician, philosopher, or computer scientist alike) who wants to understand and further pursue this timely topic.

Barwise's quest for an enrichment of the concept of set can in fact be traced back to his pioneering project, jointly with John Perry, in situation theory and situation semantics (STASS) (Barwise and Perry, 1983, p. 52):

> To carry out this project we must use, either implicitly or explicitly, a metatheory about what kinds of complexes there are and what their nature is. Like most of modern mathematics, our metatheory is a theory of sets and our complexes are set-theoretic complexes.

While Barwise and Perry had no qualms about the kind of metatheory they needed for STASS, they thought that they had to deal with the problem of large collections (Barwise and Perry, 1983, pp. 52-53):

> Our theory must distinguish between sets and collections too large to be countenanced as sets. [...] [Our] approach is to work in a metatheory like the theory KPU (Kripke-Platek admissible set-theory with urelements), one that admits an interpretation in terms of finite objects. It makes little difference here, but [this] is what we have in mind for developing the theory at a more formal level.

KPU was already the subject matter of a technical book Barwise wrote earlier (Barwise, 1975), and hence looked like a tailor-made choice. However, it was soon realized that large collections is only one of the difficulties; a more intricate issue is that of wellfoundedness. That is, assorted intuitive notions that STASS would like to explicate (e.g., some aspects of perceptual knowledge; self-awareness of the sort Descartes scrutinized; circular propositions such as the infamous Liar sentence; mutual knowledge as encountered in the Conway Paradox, etc.) cannot really be cleanly expressed in a set theory which does not admit of non-wellfounded sets. What unifies the preceding notions is their inherent circularity. Yet, historically it has invariably been the case that circularity and mathematical stringency hardly go hand in hand.[1] Put another way, The Axiom of Foundation (*Axiom der Fundierung*, FA) of classical set theory, which is equivalent to the statement that all sets are wellfounded, simply prohibits circularity.[2]

Barwise and Perry wrestled with this problem for some time, before they decided that Aczel's set theory is in fact what they were looking for.[3] In the words of Aczel (Aczel, 1988, p. 112):

> One of the most exciting areas of application for non-wellfounded sets and AFA [The Anti-Foundation Axiom] is situation semantics, a recent development of the model theoretical approach to the semantics of natural language. [...] As situations are themselves objects they can occur as components of facts. [...] So a situation can be a component of a fact that is in a situation and it is natural for circular situations to arise which contain facts about themselves. [...] While the book [*Situations and Attitudes*] does not use non-wellfounded sets they have been fully exploited in (Barwise and Etchemendy, 1987). The latter book makes use of notions of circular situation and circular proposition to discuss the Liar Paradox and uses non-wellfounded sets to represent such abstract objects.

VC consists of six parts which altogether comprise 21 chapters. (Cf. Appendix A for a coarse-grained list of contents.) Part I introduces the reader to the subject of the book in a nontechnical way and makes suggestions as to how to read it. It then summarizes in Chapter 2 the background material on classical (Zermelo-Fraenkel) set theory, ZFC, and reproduces, in a table, the axioms of $ZFC^{-}$. The system ZFC adds the Foundation Axiom, FA, to $ZFC^{-}$, and the system ZFA (which is studied in VC) adds the Anti-Foundation Axiom, AFA.

Part II, consisting of three chapters, is arguably the most readable portion of the book. Chapter 3 studies circularity in computer science, using streams, labeled transition systems, and closures (in the context of functional programming languages). It ends on an excellent discussion of self-replicating programs (Akman, 1986), which have so far been usually regarded as exotic trifles.

Circularity raises its cunning head in many parts of philosophy. The most dramatic form is in the renowed logical and semantic paradoxes, which are introduced in Chapter 4 (and then continued in Chapter 5). The Conway Paradox and its treatment via an analysis of common knowledge (mutual information), other things such as Grice's analysis of intentional activity, Descartes' celebrated dictum "Cogito, ergo sum," and Bach-Peters sentences from linguistics are elucidated as examples of circular phenomena. This chapter ends on a couple of neat exercises from probability theory where circularity is employed to some advantage.[4] Chapter 5 studies the Liar Paradox (Barwise and Etchemendy, 1987), Tarski's proposal for an infinite hierarchy of languages to resolve it, and Kripke's influential paper "Outline of a theory of truth," which demonstrated that circularity of reference is a prevalent phenomenon which may sometimes depend on non-linguistic (empirical) facts. An interesting game-theoretic paradox, popularized by Smullyan and called the Hypergame Paradox, is briefly explained. (The paradox is later resolved in Chapter 12.)

The essence of VC is Part III, which explains the universe of hypersets. AFA is presented through what is commonly known as the Solution Lemma in the literature: Every system of equations in the universe of sets has a unique solution.[5] Chapter 7 studies a fundamental question: Under what conditions do two systems of equations have the same solution? It turns out that a special relationship must hold between the parts of the two systems. This is the concept of "bisimulation," which has been rediscovered in the literature numerous times. Chapter 8 generalizes the Solution Lemma by just proving that the process of substitution used by the Lemma is well defined. Chapter 9 is devoted to a relative consistency result, viz. the proof that if ZFC is consistent, then so is ZFA.[6] It is a burly chapter and may be skipped, according to the authors, without loss of continuity.

Part IV is another highly readable part of the book, detailing the elementary applications of ZFA. In Chapter 10 a graph-theoretic version of AFA is presented, along the lines of (Aczel, 1988, Chapter 1). Readers familiar with *The Liar* (Barwise and Etchemendy, 1987, Chapter 3) will notice that this is the way hypersets are introduced in that book.

The graph alternative is an insightful way to visualize (picture) the otherwise abstract notion of a hyperset. Labeled graphs and systems of equations are really equivalent ways of representing hypersets. While the notion of a system of equations does generalize better (towards more formal work), the graph alternative is easier to discern, especially for toy examples. Chapter 11 offers a view of modal logic, showing how it might have developed if non-wellfounded sets have been available at the time the Kripkean possible world approach was taking shape. Chapter 12 is concerned with two-person games of perfect information (e.g., chess) and formalizes what it means for a competitor to have a winning strategy in such a game. The Hypergame Paradox of Smullyan is also resolved. The semantical paradoxes receive an extended treatment in Chapter 13, which is essentially a more technical amalgamation of the several arguments proposed in (Barwise and Etchemendy, 1987). While the authors note that this chapter may be skipped without loss of continuity, I would say that doing so may not be desirable; after all, this is one of the best portions of VC. A theory of streams over some alphabet is developed in Chapter 13. This chapter also gives an introduction to the important methods of coinduction and corecursion, before taking them up more formally in Part V.

Parts V and VI are not for the weak-kneed readers (such as this reviewer), for the seven chapters belonging to these parts develop and exemplify the methods of greatest fixed points, coinduction, and corecursion using difficult—occasionally impenetrable—mathematics. The only exceptions are Chapters 20 and 21 which provide some very interesting (and somewhat philosophical) views on ZFC and ZFA, and enumerate 16 open problems.[7]

What kind of reader suits to the intended audience of VC? In addition to pure mathematicians fascinated by the nifty universe of hypersets, researchers interested in using the theory in philosophy, game theory, computer science, and artificial intelligence would find VC irresistible. Note, however, that Barwise and Moss assume that the reader has enough training to follow the (generally intricate) proofs, and furthermore a more than superficial acquaintance with elementary set theory. On the other hand, as a good example of pedagogical consideration on their part, many of the instructive exercises available in VC have their solutions at the end of the book. The solutions are profitable because the exercises are not always routine; sometimes they treat material which might easily perplex an average reader.

In order to make this review a balanced one, I must unfortunately bring up a weakness of VC which has more to do with the difficulties encountered in typesetting a book of this density (in terms of mathematical notation), and presumably also with the authors' (plausible) hurry to see it in print. There are just so many typos—and few other more substantial errors, or shall we say 'conceptos'?—for a book of this size! Interestingly, Barwise and Moss have a web-based automatic bug notification form which can be used by a prospective reader to report errors—typographical or otherwise—and comments. During the period I have devoted to studying VC, I had to use this form multitudinous times. In any case, here is the web address which one should keep under her thumb when reading VC:

`http://www.phil.indiana.edu/~barwise/vccorrections.html`

In fact, I must submit that this book surely needs a quick second, corrected printing, one which will at least fix the most disturbing of these numerous typos.[8] The idea of using a web page as a dynamic—i.e., ever growing—erratum sheet is fine, but my impression is that the above page is not updated very often, the last update dating back to October, 1996. This renders it quite ineffectual after one's initial visit, despite the authors' good intentions in setting it up in the first place.

The references cited in VC do not make a lavish bibliography either, in addition to their being infested with, yes, typos. Thus, the Bibliography is yet another part of the book ready for a serious update in a second printing. Recent attempts—cf. (Pakkan and Akman, 1994; Akman and Pakkan, 1996) for two reviews—in knowledge representation and artificial intelligence advocating the use of hypersets may be suitably included. The Index, which is barely passable, unnecessarily lumps together the subject and name indices into one.

Robert Duncan, the great American poet, once solemnly declared: "I make poetry as other men make war or make love or make states or revolutions: to exercise my faculties at large" (Allen, 1960, p. 407). I think it is no exaggeration to say that in VC Barwise and Moss do the same; they make set theory to exercise their indisputable mathematical talents at large. The result is a thought-provoking book which may easily become a classic like Barwise and Etchemendy's little gem, *The Liar* (Barwise and Etchemendy, 1987).[9] It is only fitting for set theory—on many accounts, the most harmonious branch of mathematics, on a par with number theory—to be revived with a strong book of such exquisite refinement. For, in the words of Barwise and Moss (p. 64): "Circularity is here to stay; it will not be banned by fiat. [...] [T]he theory of non-wellfounded sets gives us a beautiful, coherent, and powerful tool with which to study circular phenomena in all domains."[10]

# Appendix

## A. Table of Contents

## Notes

[1] It is hard to tell why circularity has always been regarded with doubt in mathematical circles. One reason may be that circular arguments or structures are thought to be difficult to grasp, most probably due to our educational make-up in linear thinking. On a more personal note, I tend to think that popular works such as Escher's drawings may have helped to create the illusion that circularity invites nonsense.

[2] Cf. Chapter 7, especially Section 7.7, of (Copi, 1979) for an excellent discussion of Zermelo-Fraenkel set theory (ZFC), especially FA.

[3] Barwise states in his introduction to (Barwise, 1989): "In the course of this work [i.e., "Situations, sets, and the Axiom of Foundation"], I learned of Peter Aczel's work on the Anti-Foundation Axiom (AFA). This work has since had a profound impact on my own work."

[4] The interested reader may try her hand with the following little puzzle which is one of the mentioned exercises. Devise an experiment to use a fair coin to make a 3-way random decision.

[5] Cf. (Pakkan and Akman, 1995) for a practical application of the Solution Lemma, motivated by an earlier and simpler account of it given in (Barwise and Etchemendy, 1987).

[6] Since most mathematicians trust that ZFC is indeed consistent, this is a convincing result regarding the consistency of ZFA.

[7] *Caveat:* These are all difficult research problems. Barwise and Moss state that they pursued some of them but did not really obtain satisfactory solutions. So you get the picture.

[8] I know that CSLI Lecture Notes are aimed at making new results, ideas, and approaches available as *quickly* as possible—surely a laudable goal. However, this should not interfere with the usual desideratum of producing books which are free of glaring bugs.

[9] *Caveat: The Liar* is an easier book compared to VC. On the other hand, VC is a more readable book compared to Aczel's demanding treatise (Aczel, 1988) and even subsumes it in some sense.

[10] Finally, I cannot help but refer the reader to two recent books which consolidate the bright future of set theory, viz. (Devlin, 1993) and (Moschovakis, 1994). Incidentally, both (text)books include material on non-wellfounded sets.

## References

Aczel, Peter. 1988. *Non-Well-Founded Sets.* CSLI Lecture Notes, Number 14. Center for the Study of Language and Information. Stanford, California.

Akman, Varol. 1986. Writing self-replicating code. *The Computer Journal* 29: 573-574.

Akman, Varol, and Mujdat Pakkan. 1996. Nonstandard set theories and information management. *Journal of Intelligent Information Systems* 6: 5-31.

Allen, Donald M., editor. 1960. *The New American Poetry.* Grove Press, Inc. New York.

Barwise, Jon. 1975. *Admissible Sets and Structures.* Springer-Verlag. New York.

Barwise, Jon. 1989. Situations, sets, and the Axiom of Foundation. pp. 177-200 in *The Situation in Logic.* CSLI Lecture Notes, Number 17. Center for the Study of Language and Information. Stanford, California.

Barwise, Jon, and John Perry. 1983. *Situations and Attitudes.* A Bradford Book. The MIT Press. Cambridge, Massachusetts.

Barwise, Jon, and John Etchemendy. 1987. *The Liar: An Essay on Truth and Circularity.* Oxford University Press. New York.

Copi, Irving M. 1979. *Symbolic Logic.* Macmillan Publishing Co., Inc. New York.

Devlin, Keith. 1993. *The Joy of Sets: Fundamentals of Contemporary Set Theory.* Springer-Verlag. New York.

Moschovakis, Yiannis N. 1994. *Notes on Set Theory.* Springer-Verlag. New York.

Pakkan, Mujdat, and Varol Akman. 1994. Issues in Commonsense Set Theory. *Artificial Intelligence Review* 8: 279-308.

Pakkan, Mujdat, and Varol Akman. 1995. HYPERSOLVER: A graphical tool for commonsense set theory. *Information Sciences* 85: 43-61.

*Address for correspondence:*
Varol Akman,
Department of Computer Engineering and Information Science,
Bilkent University, Bilkent, Ankara 06533, Turkey.
Email: akman@cs.bilkent.edu.tr