A Hierarchy of Function Classes

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Formal Power Series

A formal power series is a sequence $f = (c_n)_{n\in\mathbb{N}}$ (indexed with the nonnegative integers) of elements of some commutative ring R. Addition and scalar multiplication are defined componentwise:

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} := (a_n + b_n)_{n\in\mathbb{N}}$$
$$\lambda(a_n)_{n\in\mathbb{N}} := (\lambda a_n)_{n\in\mathbb{N}}$$

while the multiplication is the Cauchy product:

$$(a_n)_{n\in\mathbb{N}}(b_n)_{n\in\mathbb{N}} := (\sum_{k=0}^{n} a_k b_{n-k})_{n\in\mathbb{N}}.$$

With $x := (0, 1, 0, 0, \ldots)$ we can write $f = \sum_{n\geq 0} c_n x^n$, and usually we will think of formal power series as of power series 'without requesting convergence'.
We say that $[x^n]f := c_n$ is the n^{th} coefficient of the formal series.

Formal Power Series

If the power series

$$\sum_{n \geq 0} c_n x^n$$

converges in a neighbourhood of 0, we can interpret the formal power series

$$\sum_{n \geq 0} c_n x^n$$

also as a function from that neighbourhood into R...

Classes of Formal Power Series

- Constants
- Polynomials

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- Rational Functions

$$f = \frac{1}{1 - x - x^2}$$
$$f = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

Classes of Formal Power Series

- Constants
- Polynomials
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- Algebraic Functions

$$xf^2 - f + 1 = 0$$
$$f = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \dots$$

Classes of Formal Power Series

- Constants
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- D-Finite or Holonomic Functions

$$x^2 f'' + (3x - 1)f' + f = 0$$
$$f = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + \dots$$

Classes of Formal Power Series

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- D-Finite or Holonomic Functions
- Solutions of Algebraic Differential Equations

$$x^2 f^{(iv)} f^3 + 20x^2 f''' f' f^2 + 5x f''' f^3 - 39x^2 f''^2 f^2 + 12x^2 f'' f'^2 f$$
$$-15x f'' f' f^2 + 4 f'' f^3 + 6x^2 f'^4 + 10x f'^3 f - 16 f'^2 f^2 = 0$$
$$f = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \dots$$

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- ???

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- ??? – coefficients satisfy a non-linear recurrence with polynomial coefficients

Closure Properties and Category Design

	rational	algebraic	holonomic	diff. algebraic
inv	yes	yes	no	yes
$\int$	no	no	yes	yes
Hadamard	yes	no	yes	no
subst	rational	???	algebraic	???

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What are the signatures in the category for all formal power series?

```
+: (%, %) -> %
-: % -> %
*: (%, %) -> %
D: % -> %
```

Can we be sure that multiplication and differentiation is always possible?
It seems that the categories have very little in common...

similar *toy* problems

Closure properties of subclasses of integers:

	$\mathbb{N}$	$\mathbb{Z}$	$2\mathbb{Z}+1$
-	no	yes	no
+	yes	yes	no

similar *toy* problems

Closure properties of subclasses of integers:

	$\mathbb{N}$	$\mathbb{Z}$	$2\mathbb{Z}+1$
-	no	yes	no
$+$	yes	yes	no

Closure properties of matrices:

	matrices	$\mathcal{M}_{n,m}$	$\mathcal{M}_{n,n}$
transpose	yes	no	yes
$+$	no	yes	yes
*	no	no	yes
subMatrix	yes	no	no